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Efficient Computation of Many Eigenvector Derivatives Using Dynamic Flexibility Method

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A new approach for computing eigenvector derivatives with repeated eigenvalues of a complex structural system has been successfully developed. This method utilizes a dynamic flexibility technique. Unlike other published methods, this method requires one only to solve the system linear algebraic equations once, regardless of the number of eigenvector derivatives needed to be computed. Tremendous computer time can be saved using this method. Theoretical derivation of this method is presented. Numerical examples are also presented to verify the analysis. This method can be a powerful tool for the design engineer to compute many eigenvector derivatives of a large complex structural system.

Nomenclature

A_o, A_1 ...	= (n, n) matrices to be determinated
$K \in R^{n,n}$	= real symmetric positive semidefinite stiffness matrix
$K' \in R^{n,n}$	= derivative of stiffness matrix with respect to p_j
$M \in R^{n,n}$	= real symmetric positive definite mass matrix
$M' \in R^{n,n}$	= derivative of mass matrix with respect to p_j
p_j	= jth design parameter
$R(\lambda) \in R^{n,n}$	= dynamic flexibility matrix
$Z \in R^{n,m}$	= eigenvector matrix of repeated root
$Z' \in R^{n,m}$	= derivative matrix of eigenvector of repeated root with respect to p_j
$\Lambda'_m \in R^{m,m}$	= derivative diagonal matrix of repeated eigenvalue with respect to p_j
λ_m	= repeated eigenvalue
$\Phi_h \in R^{n,h}$	= high-order eigenvector matrix excluding Φ_k
$\Phi_k \in R^{n,k}$	= lower-order eigenvector matrix including the eigenvectors Z and Φ_R
$\Phi_R \in R^{n,r}$	= rigid-body modal matrix
$\Omega_h \in R^{h,h}$	= high-order eigenvalue diagonal matrix corresponding to Φ_h
$\Omega_k \in R^{k,k}$	= lower-order eigenvalue diagonal matrix corresponding to Φ_k
$\Psi \in R^{n,m}$	= eigenvector matrix of repeated root from the generalized eigenequation

Subscripts and Superscripts

h	= number of high-order eigenvectors
k	= number of lower-order eigenvectors
m	= number of eigenvectors of repeated root λ_m
n	= number of degrees of freedom for a system
r	= number of rigid-body mode eigenvectors

Introduction

THERE are many useful methods to compute eigenvector derivatives of an eigensystem published in technical journals. These methods can be categorized into two basic groups: 1) direct computation methods and 2) indirect computation methods. Direct computation methods include the Fox and Kapoor method,[1] the Nelson method,[2] improved Nelson methods,[3,4] the Ojalvo[5] and Mills-Curran[6] methods, the Zhang and Wei direct perturbation method,[7] and the Zhang and Zhang improved direct perturbed method.[8] Direct method means that the singularity of the governing equation for eigenvector derivative [see Eq. (1)] is directly removed. Indirect computation methods include the incomplete modal methods[1,9] and the complete modal space method.[10,11] Indirect method means that the singularity of the governing equation (1) is avoided.

All aforementioned methods, except for the incomplete modal methods, need to analyze the entire system linear algebraic equations once for each different eigenvector derivative. That is, for each different eigenvalue condition, one needs to solve the system governing equations to obtain each eigenvector derivative corresponding to its eigenvalue.

For designing a large complex structural system, there are many eigenvector derivatives that need to be computed to obtain an optimal design configuration. Therefore, it is not practical for engineers to obtain the best solution because the computational efforts for an optimal structural system will be very large. The present authors have spent considerable effort trying to find the best method to minimize the computer time. Many different approaches have been studied and tried. All the results are not very promising because of either their poor numerical precision or their low computational efficiency. None of them can be considered to be as good as the method presented in this paper. This method is a new approach for computing eigenvector derivatives and uses the dynamic flexibility technique.[12] This dynamic flexibility technique was first published by Hu[12] and is modified in this paper for use in the calculation of many eigenvector derivatives. Regardless of the number of the eigenvector derivatives of interest, the dynamic flexibility method only needs to decompose the coefficient matrix of the system linear algebraic equation once. Thus this method shows high computational efficiency as compared with other methods, especially in multiple eigenvector derivative calculations of a complex design system. This method gives better numerical precision and is a much easier to perform theoretical formulation.

To show a more general approach, the analytical formulas of finding an eigenvector derivative described here are aimed at an eigensystem with repeated eigenvalues. The system with nonrepeated eigenvalue conditions can be treated as a special case.

Analytical Formulation

There are many technical papers published in the area of eigenvector derivative computation with repeated eigenvalues. A new

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method using the dynamic flexibility approach to compute many eigenvector derivatives is derived in the analysis. The dynamic flexibility method tries to find an expression equivalent to the inverse of the dynamic stiffness matrix ($K - \lambda M$). To be consistent with previously published papers, the same mathematical symbols are used for the analytical derivation as are used in other papers. The governing equation of the eigenvector derivative Z^I for an eigensystem with n degrees of freedom can be expressed as

$$(K - \lambda_m M)Z^I = MZ\Lambda_m^I - (K^I - \lambda_m M^I)Z = F \tag{1}$$

where

$$Z = \Psi\Gamma \tag{2}$$

and where Ψ is the eigenvector of repeated root λ_m obtained directly from the generalized characteristic equation

$$K\Phi = M\Phi\Omega \tag{3}$$

The term Ψ also satisfies Eq. (4):

$$\Psi^T M\Psi = I \tag{4}$$

Both matrices Λ_m^I and Γ stand for m eigenpairs of the standard characteristic equation

$$[\Psi^T(K^I - \lambda_m M^I)\Psi]\Gamma = \Gamma\Lambda_m^I \tag{5}$$

From Eq. (5), one can get

$$\Gamma^T\Gamma = I \tag{6}$$

Therefore, combining Eqs. (2), (4), and (6) yields

$$Z^T MZ = I \tag{7}$$

In regard to the existing methods,[1–8] for each different eigenvalue, one has to redecompose the coefficient matrix ($K - \lambda_m M$) as shown in Eq. (1) to compute the eigenvector derivative. If the system has large dimensions and has many eigenvector derivatives that need to be computed, the efforts for structural engineers to obtain numerical solutions become tremendously time consuming. To save computational time, a dynamic flexibility method for computing the eigenvector derivatives is developed and presented in this paper.

First Dynamic Flexibility Formula

Let us first define the dynamic flexibility matrix

$$R(\lambda) = (K - \lambda_m M)^{-1} \tag{8}$$

From Eq. (1), one can get

$$Z^I = R(\lambda)F \tag{9}$$

However, $(K - \lambda_m M)$ is a singular matrix, and thus, $R(\lambda)$ does not exist. To find a solution, the perturbation method as proposed in Ref. 7 is utilized to remove the singularity of $(K - \lambda_m M)$. Let us use the forward perturbation method to perturb about the parameter λ_m used in Eq. (1), that is,

$$\tilde{\lambda}_m^+ = \lambda_m + \eta\lambda_m = (1+\eta)\lambda_m \tag{10}$$

where η is a small quantity without dimension. Equations (8) and (9) can be rewritten as

$$R(\tilde{\lambda}^+) = [K - (1+\eta)\lambda_m M]^{-1} \tag{11}$$

$$Z_f^I = R(\tilde{\lambda}^+)F \tag{12}$$

Using the concept described in Ref. 8, one can also use the backward perturbation method to perturb about the parameter λ_m in Eq. (1), so that one can obtain

$$R(\tilde{\lambda}^-) = [K - (1-\eta)\lambda_m M]^{-1} \tag{13}$$

$$Z_b^I = R(\tilde{\lambda}^-)F \tag{14}$$

Using the complete eigenpairs $\Phi = [\Phi_k, \Phi_h]$ and $\Omega = \mathrm{diag}[\Omega_k, \Omega_h]$ of Eq. (3) as well as orthogonalization conditions $\Phi^T M\Phi = I$ and $\Phi^T K\Phi = \Omega$, one knows $\Phi^{-1} = \Phi^T M$. Thus, K and M can be expressed as

$$K = M\Phi\Omega\Phi^T M, \qquad M = M\Phi\Phi^T M \tag{15}$$

Substituting Eq. (15) into Eq. (11) yields[7,8]

$$\begin{aligned} R(\tilde{\lambda}^+) &= \{M\Phi[\Omega - (1+\eta)\lambda_m I]\Phi^T M\}^{-1} \\ &= \Phi[\Omega - (1+\eta)\lambda_m I]^{-1}\Phi^T \\ &= \Phi_k[\Omega_k - (1+\eta)\lambda_m I]^{-1}\Phi_k^T \\ &\quad + \Phi_h[\Omega_h - (1+\eta)\lambda_m I]^{-1}\Phi_h^T \end{aligned} \tag{16}$$

Equation (16) is the expansion of the dynamic flexibility matrix based on the natural vibration mode shape. Define Φ_k to consist of all lower-order mode eigenvectors that are needed to compute the derivative and can be obtained successfully from generalized eigenequation (3). The terms Φ_k contain the modes of lower and higher frequency than λ_m as well as the mode eigenvector Z of λ_m and rigid-body mode shape Φ_R. Assume $\bar{\Phi}_k \in R^{n,k1}$ (here $k1 = k - m$) to represent all lower-order mode eigenvectors excluding the eigenvector Z of λ_m. Thus one has

$$\Phi_k = [\bar{\Phi}_k, Z], \qquad \Omega_k = \mathrm{diag}[\bar{\Omega}_k, \lambda_m I] \tag{17}$$

Embedding Eq. (17) into Eq. (16) gives

$$\begin{aligned} R(\tilde{\lambda}^+) &= \bar{\Phi}_k[\bar{\Omega}_k - (1+\eta)\lambda_m I]^{-1}\bar{\Phi}_k^T - (\eta\lambda_m)^{-1}ZZ^T \\ &\quad + \Phi_h[\Omega_h - (1+\eta)\lambda_m I]^{-1}\Phi_h^T \end{aligned} \tag{18a}$$

Similarly

$$\begin{aligned} R(\tilde{\lambda}^-) &= \bar{\Phi}_k[\bar{\Omega}_k - (1-\eta)\lambda_m I]^{-1}\bar{\Phi}_k^T + (\eta\lambda_m)^{-1}ZZ^T \\ &\quad + \Phi_h[\Omega_h - (1-\eta)\lambda_m I]^{-1}\Phi_h^T \end{aligned} \tag{18b}$$

We can express the contribution of all high-order mode eigenvectors Φ_h, the third term shown in Eq. (18), as a power series of λ_m. That is, Eq. (18a) can be expressed as

$$\begin{aligned} R(\tilde{\lambda}^+) &= \sum_{i=1}^{k1}\frac{\phi_i\phi_i^T}{\omega_i - (1+\eta)\lambda_m} - \lambda_m^{-1}\eta^{-1}ZZ^T \\ &\quad + A_o + \lambda_m(1+\eta)A_1 + \lambda_m^2(1+\eta)^2A_2 + \cdots \end{aligned} \tag{19}$$

Right now $\bar{\Phi}_k$ is divided into two groups, Φ_1 and Φ_2, and their corresponding characteristic values are Ω_1 and Ω_2, respectively. Here the diagonal elements in the Ω_1 and Ω_2 matrices are separated as smaller and larger, as compared with the value of λ_m. Also, if the rigid body mode Φ_R is included in the lower-order modes $\bar{\Phi}_k$ and $m1, m2, \ldots, mm$ stand for the order numbers of m eigenvectors contained in Z, then in whole assembly, $\Phi = [\phi_1, \phi_2, \ldots, \phi_n]$ or in subassembly, $\Phi_k = [\phi_1, \phi_2, \ldots, \phi_k]$. Equation (19) can be rewritten as

$$\begin{aligned} R(\tilde{\lambda}^+) &= -\lambda_m^{-1}\eta^{-1}ZZ^T - \lambda_m^{-1}(1+\eta)^{-1}\Phi_R\Phi_R^T \\ &\quad - \sum_{i=m1-1}^{m1-1}\frac{\phi_{1i}\phi_{1i}^T}{(1+\eta)\lambda_m}\left\{1 + \frac{\omega_{1i}}{(1+\eta)\lambda_m} + \left[\frac{\omega_{1i}}{(1+\eta)\lambda_m}\right]^2 + \cdots\right\} \\ &\quad + \sum_{i=mm+1}^{k}\frac{\phi_{2i}\phi_{2i}^T}{\omega_{2i}}\left\{1 + \frac{(1+\eta)\lambda_m}{\omega_{2i}} + \left[\frac{(1+\eta)\lambda_m}{\omega_{2i}}\right]^2 + \cdots\right\} \\ &\quad + A_o + \lambda_m(1+\eta)A_1 + \lambda_m^2(1+\eta)^2A_2 + \cdots \end{aligned} \tag{20a}$$

Similarly, from Eq. (18b) one can obtain

$$R(\tilde{\lambda}-) = \lambda_m^{-1}\eta^{-1}ZZ^T - \lambda_m^{-1}(1-\eta)^{-1}\Phi_R\Phi_R^T - \sum_{i=m1-1}^{m1-1}\frac{\phi_{1i}\phi_{1i}^T}{(1-\eta)\lambda_m}\left\{1+\frac{\omega_{1i}}{(1-\eta)\lambda_m}+\left[\frac{\omega_{1i}}{(1-\eta)\lambda_m}\right]^2+\cdots\right\} + \sum_{i=mm+1}^{k}\frac{\phi_{2i}\phi_{2i}^T}{\omega_{2i}}\left\{1+\frac{(1-\eta)\lambda_m}{\omega_{2i}}+\left[\frac{(1-\eta)\lambda_m}{\omega_{2i}}\right]^2+\cdots\right\} + A_o + \lambda_m(1-\eta)A_1 + \lambda_m^2(1-\eta)^2A_2+\cdots \tag{20b}$$

The convergent speed of the power series in Eq. (20) is nearly equivalent to that of the following geometric series:

$$\frac{1}{\lambda_{k+1}-\lambda_m} = \frac{1}{\lambda_{k+1}}\left\{1+\frac{\lambda_m}{\lambda_{k+1}}+\left(\frac{\lambda_m}{\lambda_{k+1}}\right)^2+\cdots\right\} \tag{21}$$

where λ_{k+1} is the eigenvalue corresponding to $(k+1)$th eigenvector Φ_{k+1}. For this reason, it is advantageous that the number of eigenvectors contained in Φ_k is slightly more than the number of eigenvectors that are needed to compute the derivative.

From the definition of Eqs. (11) and (13), it is given that

$$[K-(1+\eta)\lambda_m M]R(\tilde{\lambda}^+) = I \tag{22a}$$

$$[K-(1-\eta)\lambda_m M]R(\tilde{\lambda}-) = I \tag{22b}$$

From Eqs. (22a) and (22b), one can get

$$\tfrac{1}{2}\{[K-(1+\eta)\lambda_m M]R(\tilde{\lambda}^+)+[K-(1-\eta)\lambda_m M]R(\tilde{\lambda}-)\} = I \tag{23}$$

Substituting Eq. (20) into Eq. (23) yields

$$\begin{aligned}&-\lambda_m^{-1}c_1K\Phi_R\Phi_R^T-\lambda_m^{-1}c_1K\Phi_1\Phi_1^T-\lambda_m^{-2}c_2K\Phi_1\Omega_1\Phi_1^T\\&-\lambda_m^{-3}c_3K\Phi_1\Omega_1^2\Phi_1^T-\cdots+K\Phi_2\Omega_2^{-1}\Phi_2^T\\&+\lambda_m K\Phi_2\Omega_2^{-2}\Phi_2^T+\lambda_m^2d_1K\Phi_2\Omega_2^{-3}\Phi_2^T+\lambda_m^3d_2K\Phi_2\Omega_2^{-4}\Phi_2^T\\&+KA_0+\lambda_mKA_1+\lambda_m^2d_1KA_2+\lambda_m^3d_2KA_3+\cdots\\&+MZZ^T+M\Phi_R\Phi_R^T+M\Phi_1\Phi_1^T+\lambda_m^{-1}c_1M\Phi_1\Omega_1\Phi_1^T\\&+\lambda^{-2}c_2M\Phi_1\Omega_1^2\Phi_1^T+\lambda_m^{-3}c_3M\Phi_1\Omega_1^3\Phi_1^T+\cdots\\&-\lambda_mM\Phi_2\Omega_2^{-1}\Phi_2^T-\lambda_m^2d_1M\Phi_2\Omega_2^{-2}\Phi_2^T-\lambda_m^3d_2M\Phi_2\Omega_2^{-3}\Phi_2^T\\&-\cdots-\lambda_mMA_0-\lambda_m^2d_1MA_1-\lambda_m^3d_2MA_2-\cdots-I=0\end{aligned} \tag{24}$$

where

$$c_1=\frac{(1+\eta)^{-1}+(1-\eta)^{-1}}{2}\qquad c_2=\frac{(1+\eta)^{-2}+(1-\eta)^{-2}}{2}$$

$$c_3=\frac{(1+\eta)^{-3}+(1-\eta)^{-3}}{2} \tag{25}$$

$$d_1=\frac{(1+\eta)^2+(1-\eta)^2}{2}\qquad d_2=\frac{(1+\eta)^3+(1-\eta)^3}{2}$$

If we collect the terms with the same power of λ_m from Eq. (24) and set them equal to zero,

$$\lambda_m^{-3}: c_3M\Phi_1\Omega_1^3\Phi_1^T-c_3K\Phi_1\Omega_1^2\Phi_1^T=0 \tag{26}$$

$$\lambda_m^{-2}: c_2M\Phi_1\Omega_1^2\Phi_1^T-c_2K\Phi_1\Omega_1\Phi_1^T=0 \tag{27}$$

$$\lambda_m^{-1}: c_1M\Phi_1\Omega_1\Phi_1^T-c_1K\Phi_1\Phi_1^T=0 \tag{28}$$

$$\lambda_m^0: KA_0+K\Phi_2\Omega_2^{-1}\Phi_2^T+M\Phi_R\Phi_R^T+M\Phi_1\Phi_1^T+MZZ^T-I=0 \tag{29}$$

$$\lambda_m^1: KA_1+K\Phi_2\Omega_2^{-2}\Phi_2^T-M\Phi_2\Omega_2^{-1}\Phi_2^T-MA_0=0 \tag{30}$$

$$\lambda_m^2: d_1KA_2+d_1K\Phi_2\Omega_2^{-3}\Phi_2^T-d_1M\Phi_2\Omega_2^{-2}\Phi_2^T-d_1MA_1=0 \tag{31}$$

$$\lambda_m^3: d_2KA_3+d_2K\Phi_2\Omega_2^{-4}\Phi_2^T-d_2M\Phi_2\Omega_2^{-3}\Phi_2^T-d_2MA_2=0 \tag{32}$$

$$\vdots\qquad\qquad\vdots$$

Combining Eqs. (26–28) results in

$$(K\Phi_1-M\Phi_1\Omega_1)\Omega_1^{q-1}\Phi_1^T=0,\qquad q=1,2,3,\ldots \tag{33}$$

Because the equation $K\Phi_1-M\Phi_1\Omega_1=0$, Eqs. (26–28) are automatically satisfied. From Eq. (29), one can obtain

$$KA_0 = I-M\tilde{\Phi}_1\tilde{\Phi}_1^T-K\Phi_2\Omega_2^{-1}\Phi_2^T \tag{34}$$

where

$$\tilde{\Phi}_1=[\Phi_R,\Phi_1,Z] \tag{35}$$

Combining Eqs. (30–32) and using relationship $K\Phi_2 = M\Phi_2\Omega_2$ give

$$KA_p = MA_{p-1},\qquad p=1,2,\ldots \tag{36}$$

Since K is a singular matrix [i.e., $\text{rank}(K)=n-r$] under the case of consideration, Eq. (34) can have a solution only under certain special conditions. The necessary and sufficient condition is

$$\Phi_R^T(I-M\tilde{\Phi}_1\tilde{\Phi}_1^T-K\Phi_2\Omega_2^{-1}\Phi_2^T)=0 \tag{37}$$

Expanding Eq. (37) gives $\Phi_R^T-\Phi_R^T=0$, that is, Eq. (34) has a solution. However, one must add r independent equations into Eq. (34) to obtain the solution. Therefore, we can consider the terms of $A_o, A_1, \ldots,$ in Eqs. (19) and (20), which are acquired from each term in the sum equation $\Phi_h(\Omega_h-\lambda_m I)^{-1}\Phi_h^T$. Thus, we can foresee Eq. (38) as

$$\Phi_R^TMA_p=0,\qquad p\geq 0 \tag{38}$$

Also

$$\Phi_R^TMA_o=0 \tag{39}$$

Obviously, from Eqs. (34) and (39), one can uniquely determine the coefficient A_o. In addition, the necessary and sufficient condition for Eq. (36) to have solutions is

$$\Phi_R^TMA_{p-1}=0,\qquad p\geq 1 \tag{40}$$

Obviously, from Eq. (38) one knows that Eq. (40) exists, that is, Eq. (36) has a solution. Therefore, Eqs. (36) and (38) can uniquely determine all the coefficients A_p $(p=1,2,\ldots)$.

After obtaining the values of coefficients $A_o, A_1, A_2, \ldots,$ and following the procedure shown in Ref. 8, the eigenvector derivative $Z\prime$ can be expressed as

$$Z\prime = \tfrac{1}{2}(Z_f+Z_b)=\tfrac{1}{2}[R(\tilde{\lambda}^+)+R(\tilde{\lambda}-)]F=R_1(\tilde{\lambda})F \tag{41}$$

where

$$R_1(\tilde{\lambda})=\tfrac{1}{2}\Phi_k\{[\Omega_k-(1+\eta)\lambda_m I]^{-1}+[\Omega_k-(1-\eta)\lambda_m I]^{-1}\}\times\Phi_k^T+A_o+\lambda_m A_1+\lambda_m^2d_1A_2+\lambda_m^3d_2A_3+\cdots \tag{42}$$

is just the first dynamic flexibility formula established by the present authors.

The idea of Ref. 8 embodied by Eq. (41) is that the term $\lambda_m^{-1}\eta^{-1}ZZ^T$ included in $R(\tilde{\lambda}^+)$ or $R(\tilde{\lambda}-)$ [see Eq. (18)] can be eliminated through the operation of $R(\tilde{\lambda}^+)+R(\tilde{\lambda}-)$. Thus, $R_1(\tilde{\lambda})$

approaches closer to exact value $R(\lambda)$ in comparison with $R(\tilde{\lambda}^+)$ or $R(\tilde{\lambda}^-)$.

Second Dynamic Flexibility Formula

In practice, Eqs. (23) and (41) imply the following approximation:

$$\begin{aligned}&\tfrac{1}{2}[(K - \tilde{\lambda}^+ M)R(\tilde{\lambda}^+) + (K - \tilde{\lambda}^- M)R(\tilde{\lambda}^-)]\\ &\quad \approx (K - \lambda_m M)[R(\tilde{\lambda}^+) + R(\tilde{\lambda}^-)]/2\\ &\quad \approx (K - \lambda_m M)R(\lambda) = I\end{aligned} \tag{43}$$

The approximation is generated by using the procedure described in Ref. 8. As mentioned earlier, the goal of using the procedure in Ref. 8 is to eliminate the second term with factor η^{-1} in Eq. (18a) or (18b), which can make $R_1(\tilde{\lambda})$ approach closer to $R(\lambda)$. The preceding goal can also be achieved by using the procedure as shown in Ref. 7. Using the present procedure results in a more simple analytical derivation.

According to Ref. 7, from the orthogonality of eigenvector Eq. (18a) is left-multiplied by $ZZ^T M$ to obtain the second term on the right-hand side of Eq. (18a) shown in Eq. (44),

$$-\lambda_m^{-1}\eta^{-1} ZZ^T = ZZ^T M R(\tilde{\lambda}^+) \tag{44}$$

Subtracting Eq. (44) from Eq. (18a) results in

$$\begin{aligned}&R(\tilde{\lambda}^+) - ZZ^T M R(\tilde{\lambda}^+) = \bar{\Phi}_k[\bar{\Omega}_k - (1+\eta)\lambda_m I]^{-1}\bar{\Phi}_K^T\\ &\quad + \Phi_h[\Omega_h - (1+\eta)\lambda_m I]^{-1}\Phi_h^T = R_2(\tilde{\lambda})\end{aligned} \tag{45}$$

Because both $R(\tilde{\lambda}^+)$ and $R_2(\tilde{\lambda})$ are symmetrical matrices, Eq. (45) can be rewritten as

$$R_2(\tilde{\lambda}) = R(\tilde{\lambda}^+)(I - MZZ^T) \tag{46}$$

Combining Eqs. (22a) and (46) yields

$$[K - (1+\eta)\lambda_m M]R_2(\tilde{\lambda}) = I - MZZ^T \tag{47}$$

Like Eqs. (19) and (20a), Eq. (45) can be rewritten as the mixed power series and can be substituted into Eq. (47). Collecting the terms with the same power of λ_m and setting them equal to zero, one can get the following:

$$\begin{aligned}&\lambda_m^{-3}: -(1+\eta)^{-3} K\Phi_1\Omega_1^2\Phi_1^T + (1+\eta)^{-3} M\Phi_1\Omega_1^3\Phi_1^T = 0\\ &\lambda_m^{-2}: -(1+\eta)^{-2} K\Phi_1\Omega_1\Phi_1^T + (1+\eta)^{-2} M\Phi_1\Omega_1^2\Phi_1^T = 0\\ &\lambda_m^{-1}: -(1+\eta)^{-1} K\Phi_1\Phi_1^T + (1+\eta)^{-1} M\Phi_1\Omega_1\Phi_1^T = 0\\ &\lambda_m^{0}: K\Phi_2\Omega_2^{-1}\Phi_2^T + KA_0 + M\Phi_R\Phi_R^T\\ &\qquad + M\Phi_1\Phi_1^T + MZZ^T - I = 0\\ &\lambda_m^{1}: (1+\eta)K\Phi_2\Omega_2^{-2}\Phi_2^T + (1+\eta)KA_1\\ &\qquad -(1+\eta)M\Phi_2\Omega_2^{-1}\Phi_2^T - (1+\eta)MA_0 = 0\\ &\lambda_m^{2}: (1+\eta)^2 K\Phi_2\Omega_2^{-3}\Phi_2^T + (1+\eta)^2 KA_2\\ &\qquad -(1+\eta)^2 M\Phi_2\Omega_2^{-2}\Phi_2^T - (1+\eta)^2 MA_1 = 0\\ &\lambda_m^{3}: (1+\eta)^3 K\Phi_2\Omega_2^{-4}\Phi_2^T + (1+\eta)^3 KA_3\\ &\qquad -(1+\eta)^3 M\Phi_2\Omega_2^{-3}\Phi_2^T - (1+\eta)^3 MA_2 = 0\\ &\quad\vdots \qquad\qquad\qquad\qquad \vdots\end{aligned} \tag{48}$$

This set of equations is, in reality, the same set of equations as shown in Eqs. (26–32). Thus both Eqs. (34) and (36) can be yielded once again by Eq. (48). After solving $A_p (p \geq 0)$ by using Eqs. (19) and (45), the second dynamic flexibility formula in this paper is

$$R_2(\tilde{\lambda}) = (I - ZZ^T M)R(\tilde{\lambda}^+) \tag{49a}$$

$$\begin{aligned}R(\tilde{\lambda}^+) &= \Phi_k[\Omega_k - (1+\eta)\lambda_m I]^{-1}\Phi_k^T + A_o\\ &\quad + \lambda_m(1+\eta)A_1 + \lambda_m^2(1+\eta)^2 A_2 + \cdots\end{aligned} \tag{49b}$$

So the eigenvector derivatives are given as

$$Z\prime = R_2(\tilde{\lambda})F \tag{50}$$

In this section, the two dynamic flexibility formulas that have been presented are approximate. But the precision of the second dynamic flexibility formula is theoretically better than that of the first formula because the first dynamic flexibility formula possesses the approximation as shown in Eq. (43).

Applications

The mathematical derivation presented in this paper is a very general approach, which includes the rigid-body modes of the system. For constrained structures, the stiffness matrix K is a nonsingular matrix. The coefficients $A_o, A_1, \ldots,$ can be independently and uniquely determined by Eqs. (34) and (36). Therefore Eqs. (34) and (35) can be rewritten as

$$KA_0 = I - M\tilde{\Phi}_1\tilde{\Phi}_1^T - K\Phi_2\Omega_2^{-1}\Phi_2^T \tag{51}$$

$$\tilde{\Phi}_1 = [\Phi_1, Z] \tag{52}$$

In regard to the derivative computation of the first-order eigenvector, Φ_1 in Eqs. (35) and (52) does not exist. When the derivative computation of the final eigenvector in Φ_k is made, Φ_2 in Eqs. (34), (51), and (54) does not exist if the number of eigenvectors contained in Φ_k is equal to the number of eigenvectors that need to compute the derivative.

To find the solution and to make the description simpler, combining Eqs. (34) and (36) yields

$$KA_p = f, \qquad p \geq 0 \tag{53}$$

In dealing with Eqs. (34) and (36), f must be expressed in a different form. That is, when solving A_o,

$$f = I - M\tilde{\Phi}_1\tilde{\Phi}_1^T - K\Phi_2\Omega_2^{-1}\Phi_2^T$$

or

$$f = I - M\tilde{\Phi}_1\tilde{\Phi}_1^T - K\Phi_2\Omega_2^{-1}\Phi_2^T \tag{54}$$

but when solving $A_p (p \geq 1)$,

$$f = MA_{p-1}, \qquad p \geq 1 \tag{55}$$

After obtaining Eq. (53), one can get a unified equation with unknown $A_p (p \geq 0)$ for a free-free system. According to the method used by Hu,[12] premultiplying $\mu M\Phi_R$ to Eq. (38) (μ is a properly chosen constant) and adding the results into Eq. (53) result in Eq. (56):

$$(K + \mu M\Phi_R\Phi_R^T M)A_p = f, \qquad p \geq 0 \tag{56}$$

or according to the method used in Ref. 1, combining Eqs. (38) and (53) gives a new set of equations,

$$\begin{bmatrix} K \\ \mu\Phi_R^T M \end{bmatrix} A_p = \begin{bmatrix} f \\ 0 \end{bmatrix}, \qquad p \geq 0 \tag{57}$$

The coefficient matrix $(K + \mu M\Phi_R\Phi_R^T M)$ of Eq. (56) can be proven to be nonsingular. The coefficient matrix shown in Eq. (57)

is a matrix with the column space of full rank. Premultiplying $[K, \mu M\Phi_R]$ to Eq. (57) yields

$$(K^2 + \mu^2 M\Phi_R\Phi_R^T M)A_p = Kf, \qquad p \geq 0 \tag{58}$$

although one has to solve the linear equation (53) or (56) once to determine the corresponding A_p for $p \geq 0$ to obtain the eigenvector derivative of the system. However, in dealing with many different eigenvectors in a given system, the constant coefficient matrix of Eq. (53) or (56) will not change and only needs to be decomposed once. Because of this, the method developed in this paper can be utilized in calculating many different eigenvector derivatives in a very efficient manner. Other methods presented in Refs. 2–7 use the dynamic stiffness $(K - \lambda_m M)$ to solve for the eigenvector derivative Z' [see Eq. (1)]. For each different eigenvalue λ_m, the coefficient matrix $(K - \lambda_m M)$ of Eq. (1) will change and needs to be re-decomposed repeatedly to compute the eigenvector derivatives corresponding to the λ_m.

Numerical Example

The numerical example used in this paper is the same example as shown in Ref. 4 with all the design parameters having the same numerical values, except that the original two beam elements has been changed to three beam elements. The example of interest is a cantilever beam with 12 degrees of freedom; each beam element has the same square cross section. Thus finite element assembly matrices of the cantilever are

$$K = \begin{bmatrix}
12 & 0 & 0 & 6 & -12 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\
 & 12 & -6 & 0 & 0 & -12 & -6 & 0 & 0 & 0 & 0 & 0 \\
 & & 4 & 0 & 0 & 6 & 2 & 0 & 0 & 0 & 0 & 0 \\
 & & & 4 & -6 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
 & & & & 24 & 0 & 0 & 0 & -12 & 0 & 0 & 6 \\
 & & & & & 24 & 0 & 0 & 0 & -12 & -6 & 0 \\
 & & & & & & 8 & 0 & 0 & 6 & 2 & 0 \\
 & & & & & & & 8 & -6 & 0 & 0 & 2 \\
 & & & & & & & & 24 & 0 & 0 & 0 \\
 & & & & & & & & & 24 & 0 & 0 \\
 & \text{symmetric} & & & & & & & & & 8 & 0 \\
 & & & & & & & & & & & 8
\end{bmatrix} \times 10^3 \tag{59}$$

$$M = \begin{bmatrix}
156 & 0 & 0 & 22 & 54 & 0 & 0 & -13 & 0 & 0 & 0 & 0 \\
 & 156 & -22 & 0 & 0 & 54 & 13 & 0 & 0 & 0 & 0 & 0 \\
 & & 4 & 0 & 0 & -13 & -3 & 0 & 0 & 0 & 0 & 0 \\
 & & & 4 & 13 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\
 & & & & 312 & 0 & 0 & 0 & 54 & 0 & 0 & -13 \\
 & & & & & 312 & 0 & 0 & 0 & 54 & 13 & 0 \\
 & & & & & & 8 & 0 & 0 & -13 & -3 & 0 \\
 & & & & & & & 8 & 13 & 0 & 0 & -3 \\
 & & & & & & & & 312 & 0 & 0 & 0 \\
 & & & & & & & & & 312 & 0 & 0 \\
 & \text{symmetric} & & & & & & & & & 8 & 0 \\
 & & & & & & & & & & & 8
\end{bmatrix} \tag{60}$$

The overall system has total of six pairs of repeated eigenvalues. The first and second frequency pairs occur separately at $\lambda_1 = 0.36346$ and $\lambda_2 = 14.365$ with eigenvectors

$$\Psi_1 = \begin{bmatrix}
0.056355 & 0 \\
0 & 0.056355 \\
0 & 0.025858 \\
-0.025858 & 0 \\
0.030823 & 0 \\
0 & 0.030823 \\
0 & 0.024588 \\
-0.024588 & 0 \\
0.009329 & 0 \\
0 & 0.009329 \\
0 & 0.016991 \\
-0.016991 & 0
\end{bmatrix} \tag{61}$$

$$\Psi_2 = \begin{bmatrix}
-0.056692 & 0 \\
0 & -0.056692 \\
0 & -0.090423 \\
0.090423 & 0 \\
0.024006 & 0 \\
0 & 0.024006 \\
0 & -0.055925 \\
0.055925 & 0 \\
0.033441 & 0 \\
0 & 0.033441 \\
0 & 0.033327 \\
-0.033327 & 0
\end{bmatrix}$$

Only the eigenvector derivatives corresponding to the first and second repeated eigenvalues are computed in this paper. The derivative calculation is made with respect to the design parameter p_j. Here p_j is chosen to be the Z axis area moment I_z of element at the free end. So derivative matrices of K and M are, respectively,

$$K' = \begin{bmatrix}
12 & 0 & 0 & 6 & -12 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 4 & -6 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
 & & & & 12 & 0 & 0 & -6 & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & 4 & 0 & 0 & 0 & 0 \\
 & & & & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & & & & 0 & 0 & 0 \\
 & \text{symmetric} & & & & & & & & & 0 & 0 \\
 & & & & & & & & & & & 0
\end{bmatrix} \times 10^3 \tag{62}$$

and $M' = 0$. Furthermore, $K'' = M'' = 0$. Solving Eq. (5) yields

$$\Lambda_1' = \begin{bmatrix} 0 & 0 \\ 0 & 0.0027581 \end{bmatrix}, \qquad \Lambda_2' = \begin{bmatrix} 0 & 0 \\ 0 & 1.8694 \end{bmatrix} \tag{63}$$

Both Γ_1 and Γ_2 corresponding to Λ_1' and Λ_2' are

$$\Gamma_1 = \Gamma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{64}$$

Table 1 Eigenvector derivatives of first repeated root with second-order approximation and six modes

Exact value		First dynamic flexibility formula		Second dynamic flexibility formula	
0	−0.52409D−3	0	−0.52409D−3	0	−0.52409D−3
0	0	0	0	0	0
0	0	0	0	0	0
0	0.10865D−2	0	0.10865D−2	0	0.10865D−2
0	0.23736D−3	0	0.23736D−3	0	0.23736D−3
0	0	0	0	0	0
0	0	0	0	0	0
0	−0.17973D−3	0	−0.17974D−3	0	−0.17973D−3
0	0.74746D−4	0	0.74746D−4	0	0.74746D−4
0	0	0	0	0	0
0	0	0	0	0	0
0	−0.13341D−3	0	−0.13341D−3	0	−0.13341D−3

Table 2 Eigenvector derivatives of second repeated root with second-order approximation and six modes

Exact value		First dynamic flexibility formula		Second dynamic flexibility formula	
0	0.66634D−2	0	0.66632D−2	0	0.66633D−2
0	0	0	0	0	0
0	0	0	0	0	0
0	−0.19372D−1	0	−0.19372D−1	0	−0.19372D−1
0	−0.35259D−2	0	−0.35259D−2	0	−0.35259D−2
0	0	0	0	0	0
0	0	0	0	0	0
0	0.12701D−1	0	0.12701D−1	0	0.12701D−1
0	0.29112D−2	0	0.29111D−2	0	0.29112D−2
0	0	0	0	0	0
0	0	0	0	0	0
0	−0.70143D−3	0	−0.70134D−3	0	−0.70133D−3

Table 3 Eigenvector derivatives of first repeated root with third-order approximation and six modes

Exact value		First dynamic flexibility formula		Second dynamic flexibility formula	
0	−0.52409D−3	0	−0.52409D−3	0	−0.52409D−3
0	0	0	0	0	0
0	0	0	0	0	0
0	0.10865D−2	0	0.10865D−2	0	0.10865D−2
0	0.23736D−3	0	0.23736D−3	0	0.23736D−3
0	0	0	0	0	0
0	0	0	0	0	0
0	−0.17973D−3	0	−0.17974D−3	0	−0.17973D−3
0	0.74746D−4	0	0.74746D−4	0	0.74746D−4
0	0	0	0	0	0
0	0	0	0	0	0
0	−0.13341D−3	0	−0.13341D−3	0	−0.13341D−3

Table 4 Eigenvector derivatives of second repeated root with third-order approximation and six modes

Exact value		First dynamic flexibility formula		Second dynamic flexibility formula	
0	0.66634D−2	0	0.66633D−2	0	0.66634D−2
0	0	0	0	0	0
0	0	0	0	0	0
0	−0.19372D−1	0	−0.19372D−1	0	−0.19372D−1
0	−0.35259D−2	0	−0.35259D−2	0	−0.35259D−2
0	0	0	0	0	0
0	0	0	0	0	0
0	0.12701D−1	0	0.12701D−1	0	0.12701D−1
0	0.29112D−2	0	0.29111D−2	0	0.29112D−2
0	0	0	0	0	0
0	0	0	0	0	0
0	−0.70143D−3	0	−0.70144D−3	0	−0.70142D−3

The new eigenvectors are

$$Z_1 = \Psi_1 \Gamma_1 = \begin{bmatrix} 0 & 0.056355 \\ 0.056355 & 0 \\ 0.025858 & 0 \\ 0 & -0.025858 \\ 0 & 0.030823 \\ 0.030823 & 0 \\ 0.024588 & 0 \\ 0 & -0.024588 \\ 0 & 0.009329 \\ 0.009329 & 0 \\ 0.016991 & 0 \\ 0 & -0.016991 \end{bmatrix} \tag{65}$$

$$Z_2 = \Psi_2 \Gamma_2 = \begin{bmatrix} 0 & -0.056692 \\ -0.056692 & 0 \\ -0.090423 & 0 \\ 0 & 0.090423 \\ 0 & 0.024006 \\ 0.024006 & 0 \\ -0.055925 & 0 \\ 0 & 0.055925 \\ 0 & 0.033441 \\ 0.033441 & 0 \\ 0.033327 & 0 \\ 0 & -0.033327 \end{bmatrix}$$

Numerical results are presented as follows.

1) The numerical results of the eigenvector derivatives of the first and second repeated eigenvalues are shown in Tables 1 and 2, respectively. Only the first six lower-order modes corresponding to $\lambda_1 \sim \lambda_3$ construct the lower-order eigenvector matrix Φ_k and are used in the computation. Results obtained from both Eqs. (42) and (49) are using the power terms up to λ^2, which is a second-order approximation.

2) Results obtained from Eqs. (42) and (49) using the power terms up to λ^3 (third-order approximation) are shown in Tables 3 and 4. Again, only the first six lower-order modes are used to construct Φ_k and used in the computation.

Also, the perturbed solution results shown in Tables 1–4 are obtained from the direct perturbation method as used in Ref. 7. That is to say, the perturbed solution is obtained by using the following formulas:

$$[K - (1+\eta)\lambda M] Z_{\text{op}}^{l} = F \tag{66a}$$

$$Z_{\text{ps}}^{l} = (I - Z Z^T M) Z_{\text{op}}^{l} \tag{66b}$$

$$Z_{\text{gs}}^{l} = Z_{\text{ps}}^{l} + ZC \tag{66c}$$

where matrix C to be determined is calculated by the procedure described in Refs. 4 and 6. In general, engineering problems normally use first-order approximation, at most use second-order approximation. This method used here can be a good tool to design complex structural systems.

The eigenvector derivatives Z^l obtained by Eqs. (42) and (49) shall be theoretically considered as the particular solution Z_{ps}^{l}. The general solution is Eq. (66c).

Finally, for the need of application and on the basis of our experience, the present authors recommend that η is about 0.001.

Conclusions

Based on technical analysis and numerical example, a new method for computing eigenvector derivatives of a large complex structural system has been successfully developed. This method uses

a dynamic flexibility technique to obtain eigenvector derivatives of an eigensystem. The conclusions obtained from numerical results are as follows.

1) The dynamic flexibility technique can be used to compute eigenvector derivatives of a complex structural system with repeated eigenvalues.

2) This method only requires one to solve the linear algebraic system equations once, regardless of the number of eigenvector derivatives needed during the design.

3) This method gives better numerical precision and an easier to perform theoretical formulation.

4) This method gives the mathematical expression for eigenvector derivatives and is numerically stable.

5) This method can be a very useful tool for a design engineer to compute eigenvector derivation of a large complex structural system.

References

[1]Fox, R. L., and Kapoor, M. P., "Rate of Change of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 6, No. 12, 1968, pp. 2426–2429.

[2]Nelson, R. B., "Simplified Calculation of Eigenvector Derivatives," *AIAA Journal*, Vol. 14, No. 9, 1976, pp. 1201–1205.

[3]Ojalve, I. U., "Gradient for Large Structural Models with Repeated Frequencies," Society of Automotive Engineers, Paper 86-1789, Warrendale, PA, Oct. 1986.

[4]Dailey, R. L., "Eigenvector Derivatives with Repeated Eigenvalues," *AIAA Journal*, Vol. 27, No. 4, 1989, pp. 486–491.

[5]Ojalvo, I. U., "Efficient Computation of Mode-Shape Derivatives for Large Dynamic Systems," *Proceedings of the AIAA 27th Structures, Structural Dynamics, and Materials Conference*, Pt. 2, AIAA, New York, 1986, pp. 242–247.

[6]Mills-Curran, W. C., "Calculation of Eigenvector Derivatives for Structures with Repeated Eigenvalues," *AIAA Journal*, Vol. 26, No. 7, 1988, pp. 867–871.

[7]Zhang, D. W., and Wei, F. S., "Direct Perturbation Methods for Calculation of Eigenvector Derivatives with Repeated Eigenvalues," *Acta Mechanica Solida Sinica*, Vol. 14. No. 4, 1993, pp. 337–341.

[8]Zhang, D. W., and Zhang, L. M., "Improved Direct Perturbed Methods for Calculation of Eigenvector Derivatives," *Computational Structural Mechanics and Applications*, Vol. 10, No. 4, 1993, pp. 449–455.

[9]Wang, B. P., "Improved Approximate Methods for Computing Eigenvector Derivatives in Structural Dynamics," *AIAA Journal*, Vol. 29, No. 6, 1991, pp. 1018–1020.

[10]Zhang, D. W., and Wei, F. S., "Complete Modal Method for Eigenvector Derivatives with Repeated Eigenvalues," *Acta Mechanica Solida Sinica*, Vol. 13, No. 4, 1992, pp. 347–352.

[11]Zhang, D. W., and Wei, F. S., "Computation of Eigenvector Derivatives with Repeated Eigenvalues Using a Complete Modal Space," *AIAA Journal*, Vol. 33, No. 9, 1995, pp. 1749–1753.

[12]Hu, H. C., "Natural Vibration Problem of a System with Many Degrees of Freedom (Fixed-Interface Modal Synthesis)," *Acta Aeronautica et Astronautica Sinica*, Vol. 1, No. 2, 1980, pp. 28–35.

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